Characterizing Extremal Elements under the Partial Order on Preference Profiles

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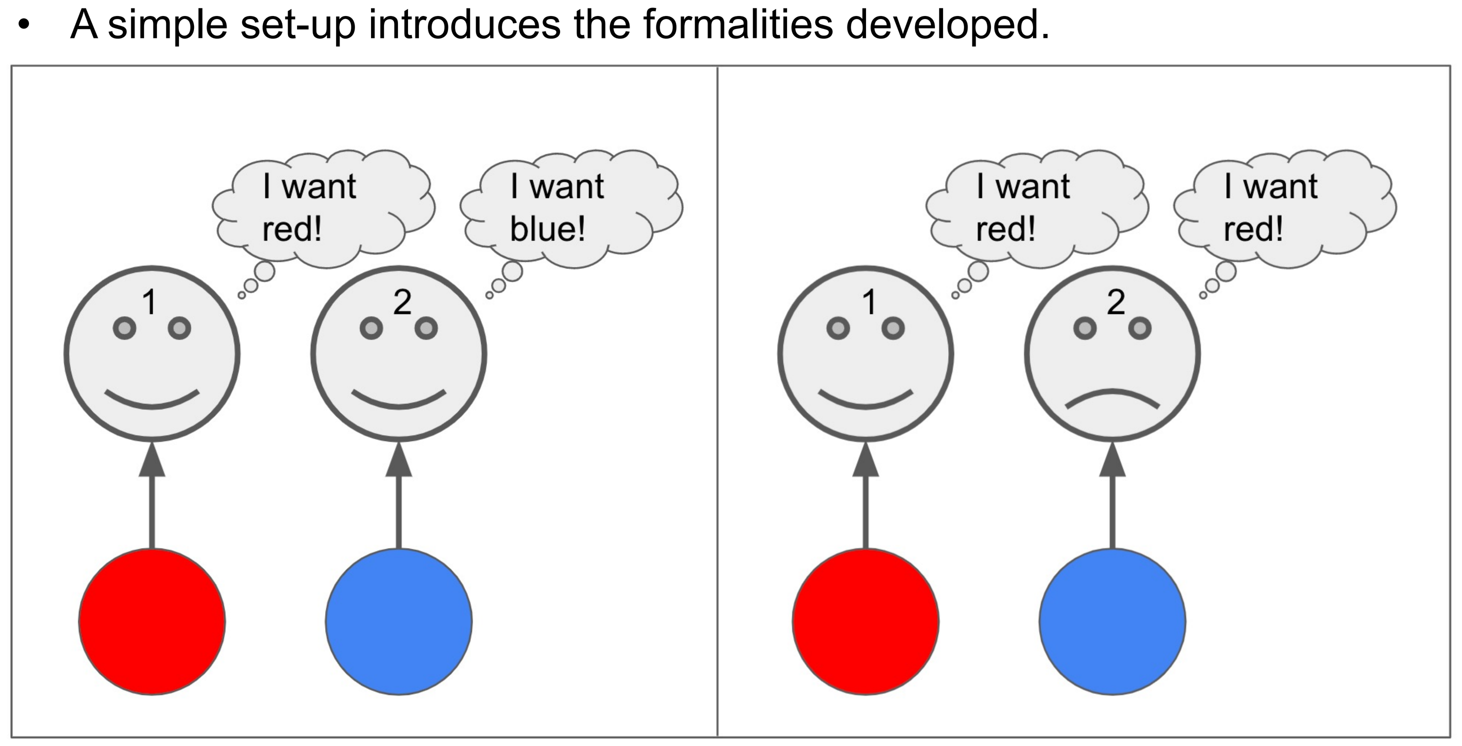
Motivation

- Intuition allows us to distinguish between "good" and "bad" preferences. A harmonious network of preferences will satisfy the needs of more individuals after allocation. On the other hand, preferences such as discrimination cause incompatibility, leading to conflict regardless of allocation strategy.
- A mathematical order imposed on preference profiles allows economists to rank complicated networks of preferences. It provides insight on the relative harmony of networks of preferences.
- Quantifying preference profiles on a ranking from "bad" to "good" provides insight into how preferences can change in order to become more harmonious and produce more satisfaction.

Two Individual – Two Allocation Basis

- A simple set-up introduces the formalities developed.



Two preference profiles depicted for a two individual, two allocation set-up.

- Two different objects must be allocated to two different individuals. The allocation space denoted X contains all possible allocations.

$$X := \{(r,b),(b,r)\}$$

$\rightarrow$ "(r,b)": 1 gets red pen; 2 gets blue pen.
$\rightarrow$ "(b,r)": 1 gets blue pen; 2 gets red pen.

- Preference profiles denoted P are collections of preferences for a group. Each box above presents a different preference profile.

1 prefers red while 2 prefers blue $\Rightarrow$

$$P: \quad (r,b) \succ_{P_1} (b,r), \quad (r,b) \succ_{P_2} (b,r),$$

both 1 and 2 prefer red $\Rightarrow$

$$P': \quad (r,b) \succ_{P_1'} (b,r), \quad (b,r) \succ_{P_2'} (r,b).$$

- A Pareto efficient (PE) allocation is one such that no allocation can be changed to make an individual better off without making another individual worse off. A Pareto frontier is the collection of PE allocations.

$$PE(P) = \{(r,b)\}. \qquad PE\left(P'\right) = \{(r,b),(b,r)\}.$$

- A ranking vector inputs an allocation and preference profile, then outputs a vector of rankings indexed by individuals. A low ranking indicates that an individual prefers the input allocation given the preference profile.

$$R_P(r,b) = (1,1) \qquad R_{P'}(r,b) = (1,2) \quad R_{P'}(b,r) = (2,1)$$

- Comparison of ranking vectors allow us to determine an order for preference profiles. Lower ranking preference profiles are called more harmonious.

$$R_P(r,b) = (1,1) \lneqq \begin{cases} R_{P'}(r,b) = (1,2) \\ R_{P'}(b,r) = (2,1) \end{cases}$$

Generalization of Ranking Vector

- Beyond the Two-Individual, Two-Allocation example, the complexity of preferences forces a more formal definition of the ranking vector. P_i denotes a preference profile for individual *i*. *x* is the input allocation.

$$R_{P_i}(x) := 1 + \#\{z \in X : z \succ_{P_i} x\}$$

- The ranking of allocation *x* for an individual *i* under preference profile *P* increases according to the number of allocations preferred over *x*.

Partial Order on Preference Profiles

Given (X, P), we write

$$P \trianglerighteq P'$$

if there exists an *onto* mapping

$$\psi : PE\left(P'\right) \to PE(P)$$

such that

$$R_P(\psi(x)) \leq R_{P'}(x), \quad \text{for every } x \in PE\left(P'\right).$$

Maximal Elements of the Partial Order

- A Preference Profile is maximally harmonious when there is perfect agreement on the best allocation. This leads to a full set of identity ranking vectors.

$$R_{\overline{P}}(x) = \mathbf{1}_N,$$

- The proof regarding the characterization of the maximally harmonious preference profile is trivial.

Minimal Elements of the Partial Order

If P is a $\trianglerighteq$-minimal element if and only if P is strict and such that every allocation in X is Pareto efficient under P.

$$\text{PE}(P) = X$$

- Proving the following characterization of the minimal ("worst") preference profile requires three intermediate transitions.

Lemma 1: All allocations must be Pareto efficient.

Lemma 2: No two individuals can be indifferent between an allocation that is Pareto efficient and another allocation.

Lemma 3: No individual can be indifferent between an allocation that is Pareto efficient and another allocation.

Proving the Lemmas of Minimality

Lemma 1: If $PE_X(P) \neq X$, then P is not $\trianglerighteq$-minimal

- Assume P is minimally harmonious and has allocations which are not Pareto efficient.
- Let $\bar{x}$ be an allocation that Pareto dominates $\underline{x}$ and is preferred by individual *i* over $\underline{x}$.
- Construct an alternative preference profile *P'* that is identical to *P*, except individual *i* now prefers $\underline{x}$ over $\bar{x}$. Under this preference profile, $\underline{x}$ becomes Pareto efficient.
- There exists an onto mapping between the Pareto frontier of *P'* and *P* such that the *P'* ranking vector of all Pareto efficient allocations is weakly better than the *P* ranking vector of all mapped Pareto efficient allocations.
- By finding a less harmonious preference profile *P*, contradict the first statement and prove Lemma 1.

Lemma 2: If $\exists\, i, j \in N$ and $x \in PE_X(P)$ such that $x \sim_{P_i} y$ and $x \sim_{P_j} y$ then P is not $\trianglerighteq$-minimal.

- Assume *P* is minimally harmonious and has allocations that two individuals are indifferent between.
- Perturb the indifference between allocations in a new preference profile *P'* such that both individuals prefer a single object.
- Demonstrate that *P'* is less harmonious than *P* by showing the existence of an onto mapping between Pareto frontiers.
- By finding a less harmonious preference profile, contradict the first statement and prove Lemma 2.

Lemma 3: If $\exists\, i \in N$ and $x \in PE_X(P)$ such that $x \sim_{P_i} y$, then P is not $\trianglerighteq$-minimal

- Assume *P* is minimally harmonious and has allocations that one individual is indifferent between.
- Perturb the indifference between allocations in a new preference profile *P'* such that the individual's indifference is now a strict preference.
- Demonstrate that *P'* is less harmonious than *P* by showing the existence of an onto mapping between Pareto frontiers. To do this, Lemma 2 is used.
- By finding a less harmonious preference profile, contradict the first statement and prove Lemma 3.