Discrete Morse Theory and Finite Topological Spaces

Adam Zheleznyak (SAS '22)

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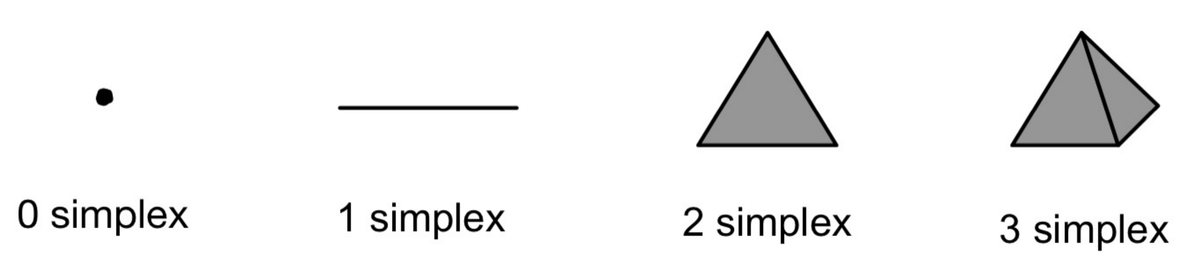
Introduction

Over the summer I investigated the potential connections between two topics in mathematics: discrete Morse theory and finite topological spaces.

The reason it is worth looking at how these two areas are related is because finite topological spaces can be analyzed using structures called *simplicial complexes*, and discrete Morse theory was specifically developed to study these objects. Simplicial complexes are spaces that are formed from n-simplices, which can be thought of as generalized triangles. Here are some lower dimensional examples:

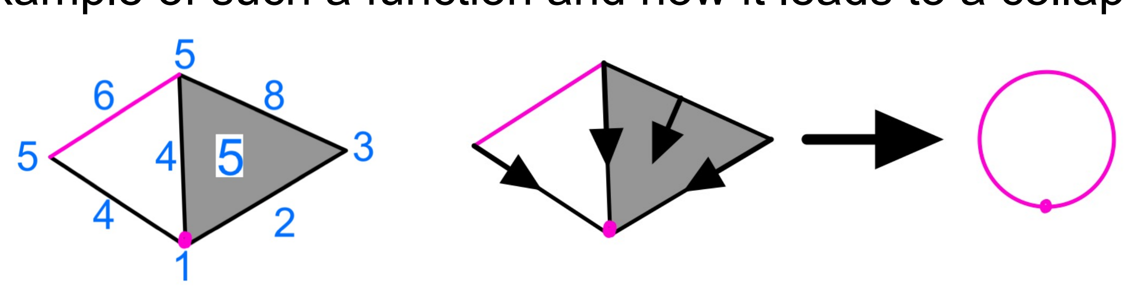


And here are two examples of simplicial complexes:

Discrete Morse Theory

Discrete Morse theory is a relatively recent development (mid 1990s) which can be thought of as a combinatorial analog to (smooth) Morse theory, which is where one analyzes the topology of differentiable manifolds using special smooth functions.

In discrete Morse theory, you study the topology of a simplicial complex by defining a *discrete Morse function* on its simplices. Using this function, you can obtain a *gradient vector field* which tells you how you can collapse your simplicial complex to get a new space that is effectively the same (*homotopy equivalent*). Often this simplifies the space and makes it easier or more efficient to study. Here is an example of such a function and how it leads to a collapse:



Applications of Discrete Morse Theory

Discrete Morse theory has applications in areas like combinatorics, topological data analysis, and computer science. This stems from the fact that it helps us find smaller spaces that are equivalent to the ones we care about.

For example, given a topological data set in which we have many data points scattered around a space, we can connect the points that are close to each other to form a simplicial complex. Then we can take this simplicial complex and analyze the topology to gain information about the geometry of our data (this is called persistent homology). Discrete Morse theory can be used in such an algorithm to make it much more efficient.

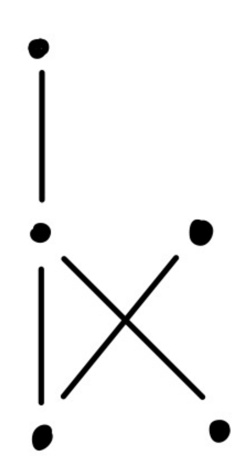
Finite Topological Spaces

A *finite topological space* is a space that contains a finite number of points. This is different than the spaces mathematicians typically study (for example, spaces like the real plane, a circle, or a torus all contain infinite points).

However, despite only containing a finite number of points, the topology of these spaces can actually be quite interesting. In fact, given most well-behaved spaces (all CW-complexes specifically), there is a finite topological space that is weakly equivalent to it (*weak homotopy equivalent*). For example, a circle can be modeled using only four points, and a sphere with six points!

In addition, there is a way to get a simplicial complex that is weakly equivalent to a given finite topological space, and vice versa.

Finite topological spaces that satisfy something called the T_0 property are particularly interesting because they can be represented using a *partially ordered set*. In fact, all partially ordered sets represent a unique finite topological space. This makes it easier to think about the space and lets us use tools from combinatorics. Here is an example of a partially ordered set representing a space with five points:



Connections

Using the same ideas developed in discrete Morse theory, one can develop a discrete Morse theory on certain classes of finite topological spaces.

For the theory to apply, we need our space to satisfy certain properties which, in a simplified sense, says that our finite topological space is formed by added spheres of increasing dimension.

Given that we have these properties, we can define a discrete Morse function on our finite topological space and find something equivalent to a gradient vector field. This lets us use the same methods as before to find a simpler space that is equivalent to our original one. The idea is that this can help us think about or calculate things relating to our finite topological space.

Unfortunately, the properties are somewhat strict, but there are still certain classes of spaces that satisfy them. For example, there are spaces called *finite manifolds* that might be studied in combinatorics.

References

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